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Collisions of optical ultra-short vector pulses

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Abstract
Asymptotic two-pulse solutions of a modified (via attaching a nonlinear derivative term) vector nonlinear Schrödinger equation (MVNSE) are analyzed using the Hirota (bilinearization) method. This equation is known to describe the propagation of ultra-short optical pulses in nonlinear fibers (in particular, in the optical-crystal fibers). Some pulse-parameter regimes of applicability of the solutions to study the pulse collisions are estimated. It is found, for these regimes, that the collision of ultra-short vector pulses of almost equal velocities results in a transformation of their polarizations similar as that of the Manakov solitons whenever the pulse-interaction effects can be neglected.

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A modified nonlinear Schrödinger equation (MNSE) has been studied for over one decade with the application to the coherent propagation of ultra-short (subpicosecond) optical pulses [1, 2]. Earlier, it was used to analyze the self-steepening (optical shock formation) effect which is not an inherent property of the dynamics of the ultra-short pulses however [3]. The importance of the ultra-short pulse research has grown since the production of the photonic-crystal nonlinear fibers began. Such fibers enable much higher compression of the optical pulses than the conventional fibers [4, 5]. The modified vector nonlinear Schrödinger equation (MVNSE)

$$\mathrm{i}q_{j,t} + q_{j,xx} + \mu\left(\sum_{k=1}^{2}|q_k|^2\right)q_j + \mathrm{i}\gamma\left[\left(\sum_{k=1}^{2}|q_k|^2\right)q_j\right]_{,x} = 0, \qquad (1)$$

($j = 1, 2$) is the proper equation of motion of the ultra-short pulse envelope (including the polarization effects) for the case of random birefringence of the optical fiber (applicable also to the case of a specific (non-random) nonlinear birefringence) [1, 6]. The MVNSE (with $\mu = 0$) is also useful in the plasma physics, where it describes polarized Alfven waves [7]. Unfortunately, in contrast to the scalar MNSE [8], one did not manage to find explicit multi-soliton solutions of its vector counterpart.

On the other hand, collisions of usual (longer) optical pulses, vector solitons described by the vector nonlinear Schrödinger equation (VNSE), result in changing their polarizations with dependence on initial polarizations $\boldsymbol{c}_{(1)}, \boldsymbol{c}_{(2)}$ velocities $4\zeta_1', 4\zeta_2'$ and widths $1/\zeta_1'', 1/\zeta_2''$ of the colliding solitons following the Manakov relation, [9],

$$\begin{aligned} \boldsymbol{c}_{(1)}' &= \frac{1}{\chi}\frac{\zeta_1^* - \zeta_2}{\zeta_1^* - \zeta_2^*}\left[\boldsymbol{c}_{(1)} + \frac{\zeta_2 - \zeta_2^*}{\zeta_2^* - \zeta_1^*}(\boldsymbol{c}_{(2)}^* \cdot \boldsymbol{c}_{(1)})\boldsymbol{c}_{(2)}\right], \\ \boldsymbol{c}_{(2)}' &= \frac{1}{\chi}\frac{\zeta_1^* - \zeta_2}{\zeta_1 - \zeta_2}\left[\boldsymbol{c}_{(2)} + \frac{\zeta_1 - \zeta_1^*}{\zeta_2 - \zeta_1}(\boldsymbol{c}_{(1)}^* \cdot \boldsymbol{c}_{(2)})\boldsymbol{c}_{(1)}\right], \end{aligned} \tag{2}$$

where $\zeta_j \equiv \zeta_j' + \mathrm{i}\zeta_j''$, and

$$\chi = \frac{|\zeta_1 - \zeta_2^*|}{|\zeta_1 - \zeta_2|}\left[1 + \frac{(\zeta_1 - \zeta_1^*)(\zeta_2^* - \zeta_2)}{|\zeta_1 - \zeta_2|^2}|\boldsymbol{c}_{(1)}^* \cdot \boldsymbol{c}_{(2)}|^2\right]^{1/2}. \tag{3}$$

Recently, this effect has been observed, [10], and utilized in different concepts of the information processing via collisions of vector solitons [11, 12]. The ultra-short pulses propagating in the photonic-crystal fibers seem to be especially useful for data processing and long-distance communication because of low losses of the intensity, thus, a long-time coherence [4]. Since the one-pulse solutions of the MVNSE have the simple form of the MNSE solution multiplied by a constant (unit) polarization vector, one is interested if the collisions of the ultra-short pulses result in similar polarization transformation (without changing the pulse velocities and widths) as that induced by the Manakov-soliton collisions. In the present paper, I give a partial answer to this point relevant to a specific regime of parameters of the colliding pulses. For small differences of the velocities and widths of the colliding pulses (big enough, however, to exclude the pulse-interaction effects) and for small pulse velocities, I predict that the one-hump solutions of the MVNSE transform their polarizations during the collision following a rule similar to the Manakov one.

Looking for two-pulse solutions of the MVNSE let us perform a decomposition of this equation following the Hirota (bilinearization) method [13]. Assuming that

$$q_1 = gf^*/f^2, \qquad q_2 = hf^*/f^2, \tag{4}$$

one finds the MVNSE to be equivalent to the system of equations

$$\left(\mathrm{i}D_t + D_x^2\right)g \cdot f = 0, \quad \left(\mathrm{i}D_t + D_x^2\right)h \cdot f = 0, \tag{5a}$$

$$D_x f^* \cdot f + \frac{\mathrm{i}\gamma}{2}(|g|^2 + |h|^2) = 0, \tag{5b}$$

$$\left(\mathrm{i}D_t + D_x^2\right)f^* \cdot f - \mu(|g|^2 + |h|^2) + \mathrm{i}\gamma(D_x g^* \cdot g + D_x h^* \cdot h) = 0, \tag{5c}$$

$$f^* D_x^2 f \cdot f - \mu f(|g|^2 + |h|^2) - \mathrm{i}\gamma(g^* D_x g \cdot f + h^* D_x h \cdot f) = 0. \tag{5d}$$

Here, D_t, D_x denote Hirota operators of differentiation over the time and position, respectively, defined by

$$D_t^m D_x^n b(x,t) \cdot c(x,t) \equiv (\partial/\partial t - \partial/\partial t')^m (\partial/\partial x - \partial/\partial x')^n b(x,t)c(x',t')|_{x=x',t=t'}. \tag{6}$$

Although equation (5*d*) is the trilinear one, we look for its solution in the form of the Hirota expansion relevant to solutions of bilinear equations (similar as in the cases of the decompositions of, e.g. Getmanov equation or Landau–Lifshitz equation [13, 14]).

Let us note that, unlike for the scalar MNSE, the bilinearization (5*a*)–(5*d*) does not lead to an equation equivalent to the MVNSE for $q_j' = q_j f/f^*$,

$$\mathrm{i}q_{j,t}' + q_{j,xx}' + \mu\left(\sum_{k=1}^{2}|q_k'|^2\right)q_j' + \mathrm{i}\gamma\left(\sum_{k=1}^{2}|q_k'|^2\right)q_{j,x}' \neq 0, \tag{7}$$

(the scalar (gauge) transformed MNSE was solved (for $\mu = 0$) in [15]).

We find the exact two-pulse solution of the system (5*b*)–(5*d*) in the form

$$\begin{aligned} f &= 1 + \mathrm{e}^{\eta_1+\eta_1^*+R_1}(\mu + \mathrm{i}\gamma k_1) + \mathrm{e}^{\eta_2+\eta_2^*+R_2}(\mu + \mathrm{i}\gamma k_2) + \mathrm{e}^{\eta_1+\eta_2^*+\delta_0}(\mu + \mathrm{i}\gamma k_1) \\ &\quad + \mathrm{e}^{\eta_2+\eta_1^*+\delta_0^*}(\mu + \mathrm{i}\gamma k_2) + \mathrm{e}^{\eta_1+\eta_2+\eta_1^*+\eta_2^*+R}(\mu + \mathrm{i}\gamma k_1)(\mu + \mathrm{i}\gamma k_2), \\ g &= \alpha_1\,\mathrm{e}^{\eta_1} + \alpha_2\,\mathrm{e}^{\eta_2} + \mathrm{e}^{\eta_1+\eta_2+\eta_2^*+\delta_2}(\mu - \mathrm{i}\gamma k_2^*) + \mathrm{e}^{\eta_1+\eta_2+\eta_1^*+\delta_1}(\mu - \mathrm{i}\gamma k_1^*), \\ h &= \beta_1\,\mathrm{e}^{\eta_1} + \beta_2\,\mathrm{e}^{\eta_2} + \mathrm{e}^{\eta_1+\eta_2+\eta_2^*+\delta_2'}(\mu - \mathrm{i}\gamma k_2^*) + \mathrm{e}^{\eta_1+\eta_2+\eta_1^*+\delta_1'}(\mu - \mathrm{i}\gamma k_1^*), \end{aligned} \tag{8}$$

where

$$\begin{aligned} \mathrm{e}^{R_j} &\equiv \frac{|\alpha_j|^2 + |\beta_j|^2}{2(k_j + k_j^*)^2}, \qquad \mathrm{e}^{\delta_0} \equiv \frac{(\alpha_1\alpha_2^* + \beta_1\beta_2^*)}{2(k_1 + k_2^*)^2}, \\ \mathrm{e}^{R} &\equiv \frac{|k_2 - k_1|^2}{4(k_1 + k_1^*)(k_2 + k_2^*)|k_1 + k_2^*|^2} \\ &\quad \times \left[\frac{(|\alpha_1|^2 + |\beta_1|^2)(|\alpha_2|^2 + |\beta_2|^2)}{(k_1 + k_1^*)(k_2 + k_2^*)} - \frac{(\alpha_1\alpha_2^* + \beta_1\beta_2^*)(\alpha_2\alpha_1^* + \beta_2\beta_1^*)}{|k_1 + k_2^*|^2}\right], \\ \mathrm{e}^{\delta_1} &\equiv \frac{(k_2 - k_1)}{2(k_1 + k_1^*)(k_2 + k_1^*)}\left[\frac{\alpha_2(|\alpha_1|^2 + |\beta_1|^2)}{k_1 + k_1^*} - \frac{\alpha_1(\alpha_2\alpha_1^* + \beta_2\beta_1^*)}{k_2 + k_1^*}\right], \\ \mathrm{e}^{\delta_2} &\equiv \frac{(k_1 - k_2)}{2(k_2 + k_2^*)(k_1 + k_2^*)}\left[\frac{\alpha_1(|\alpha_2|^2 + |\beta_2|^2)}{k_2 + k_2^*} - \frac{\alpha_2(\alpha_1\alpha_2^* + \beta_1\beta_2^*)}{k_1 + k_2^*}\right], \\ \mathrm{e}^{\delta_1'} &\equiv \frac{(k_2 - k_1)}{2(k_1 + k_1^*)(k_2 + k_1^*)}\left[\frac{\beta_2(|\alpha_1|^2 + |\beta_1|^2)}{k_1 + k_1^*} - \frac{\beta_1(\alpha_2\alpha_1^* + \beta_2\beta_1^*)}{k_2 + k_1^*}\right], \\ \mathrm{e}^{\delta_2'} &\equiv \frac{(k_1 - k_2)}{2(k_2 + k_2^*)(k_1 + k_2^*)}\left[\frac{\beta_1(|\alpha_2|^2 + |\beta_2|^2)}{k_2 + k_2^*} - \frac{\beta_2(\alpha_1\alpha_2^* + \beta_1\beta_2^*)}{k_1 + k_2^*}\right] \end{aligned} \tag{9}$$

that is similar to the two-soliton solution of the (scalar) MNSE by Liu and Wang [8] and to the two-soliton solution of the VNSE [16]. Here $\eta_{1(2)} \equiv k_{1(2)}(x + \mathrm{i}k_{1(2)}t) + \eta_{o1(2)}$, $\mathrm{Re}\, k_{1(2)}$ is related to the inversion of the first-pulse (second-pulse) width, while $\mathrm{Im}\, k_{1(2)}$ is related to the first-pulse (second-pulse) velocity, and $\eta_{o1(2)}$ denotes a constant connected to the initial phase and position of the pulse. Inserting these functions into the left-hand sides of (5*a*), we arrive at

$$\begin{aligned} \left(\mathrm{i}D_t + D_x^2\right)g\cdot f &= \mathrm{i}\gamma(k_2 - k_1)(\alpha_1\beta_2 - \alpha_2\beta_1)(\mathrm{e}^{\eta_1+\eta_2+\eta_1^*}\beta_1^* + \mathrm{e}^{\eta_1+\eta_2+\eta_2^*}\beta_2^*) \\ &\quad + \mathrm{e}^{2\eta_1+\eta_2+\eta_1^*+\eta_2^*}\frac{\gamma^2 k_1|k_2 - k_1|^2(\alpha_1\beta_2 - \alpha_2\beta_1)}{2(k_1 + k_1^*)^2(k_1 + k_2^*)^2} \\ &\quad \times [k_1\alpha_1(\alpha_2^*\beta_1^* - \alpha_1^*\beta_2^*) + k_1^*\beta_1^*(\alpha_1\alpha_2^* + \beta_1\beta_2^*) - k_2^*\beta_2^*(|\alpha_1|^2 + |\beta_1|^2)] \\ &\quad + \mathrm{e}^{\eta_1+2\eta_2+\eta_1^*+\eta_2^*}\frac{\gamma^2 k_2|k_2 - k_1|^2(\alpha_1\beta_2 - \alpha_2\beta_1)}{2(k_2 + k_1^*)^2(k_2 + k_2^*)^2} \\ &\quad \times [k_2\alpha_2(\alpha_2^*\beta_1^* - \alpha_1^*\beta_2^*) - k_2^*\beta_2^*(\alpha_2\alpha_1^* + \beta_2\beta_1^*) + k_1^*\beta_1^*(|\alpha_2|^2 + |\beta_2|^2)], \\ \left(\mathrm{i}D_t + D_x^2\right)h\cdot f &= \mathrm{i}\gamma(k_1 - k_2)(\alpha_1\beta_2 - \alpha_2\beta_1)(\mathrm{e}^{\eta_1+\eta_2+\eta_1^*}\alpha_1^* + \mathrm{e}^{\eta_1+\eta_2+\eta_2^*}\alpha_2^*) \\ &\quad + \mathrm{e}^{2\eta_1+\eta_2+\eta_1^*+\eta_2^*}\frac{\gamma^2 k_1|k_2 - k_1|^2(\alpha_1\beta_2 - \alpha_2\beta_1)}{2(k_1 + k_1^*)^2(k_1 + k_2^*)^2} \\ &\quad \times [k_1\beta_1(\alpha_2^*\beta_1^* - \alpha_1^*\beta_2^*) - k_1^*\alpha_1^*(\alpha_1\alpha_2^* + \beta_1\beta_2^*) + k_2^*\alpha_2^*(|\alpha_1|^2 + |\beta_1|^2)] \\ &\quad + \mathrm{e}^{\eta_1+2\eta_2+\eta_1^*+\eta_2^*}\frac{\gamma^2 k_2|k_2 - k_1|^2(\alpha_1\beta_2 - \alpha_2\beta_1)}{2(k_2 + k_1^*)^2(k_2 + k_2^*)^2} \\ &\quad \times [k_2\beta_2(\alpha_2^*\beta_1^* - \alpha_1^*\beta_2^*) + k_2^*\alpha_2^*(\alpha_2\alpha_1^* + \beta_2\beta_1^*) - k_1^*\alpha_1^*(|\alpha_2|^2 + |\beta_2|^2)] \end{aligned} \tag{10}$$

and see that the right-hand sides of (10) are equal to zero for the specific cases: when the polarizations of both the pulses are identical ($\alpha_1\beta_2 - \alpha_2\beta_1 = 0$, then the solution can be found by studying the simpler (scalar) MNSE), or when the velocities and widths of the pulses are equal ($k_1 = k_2$). In a general case, for $\mathrm{Re}\, k_j\, \mathrm{Im}\, k_j < 0$, the right-hand sides of (10) tend to zero with $t \to -\infty$. Thus, we have found the asymptotic two-pulse solution of the MVNSE (relevant when both the pulses are far away from each other) as the limit of the functions (8)

$$\begin{aligned} q_1 &= gf^*/f^2 = \alpha_1 q^{(1)} + \alpha_2 q^{(2)}, \\ q_2 &= hf^*/f^2 = \beta_1 q^{(1)} + \beta_2 q^{(2)}, \\ q^{(j)} &\equiv \frac{\mathrm{e}^{\eta_j}[1 + \mathrm{e}^{\eta_j+\eta_j^*+R_j}(\mu - \mathrm{i}\gamma k_j^*)]}{[1 + \mathrm{e}^{\eta_j+\eta_j^*+R_j}(\mu + \mathrm{i}\gamma k_j)]^2}. \end{aligned} \tag{11}$$

Denoting some pulse parameters as: x_{oj} (the initial position, $x_{oj} = -(\mathrm{Re}\,\eta_{oj} + R_j/2)/\mathrm{Re}\,k_j$), v_j (the velocity, $v_j = 2\,\mathrm{Im}\,k_j \equiv 4\zeta_j'$), ζ_j'' (the width inversion, $\zeta_j'' = \mathrm{Re}\,k_j/2$), the pulse envelopes can be written in the form

$$\begin{aligned} (\alpha_j, \beta_j)q^{(j)}(x,t) &= (c_{(j)1}, c_{(j)2})\mathrm{i}4 \cdot 2^{1/2}\zeta_j''\mathrm{e}^{\mathrm{i}2\zeta_j'x + \mathrm{i}4(\zeta_j''^2 - \zeta_j'^2)t}\mathrm{e}^{P_j^*/2 - P_j} \\ &\quad\times \frac{\mathrm{e}^{-2\zeta_j''(x - x_{oj} - 4\zeta_j't) - P_j^*/2} + \mathrm{e}^{2\zeta_j''(x - x_{oj} - 4\zeta_j't) + P_j^*/2}}{[\mathrm{e}^{-2\zeta_j''(x - x_{oj} - 4\zeta_j't) - P_j/2} + \mathrm{e}^{2\zeta_j''(x - x_{oj} - 4\zeta_j't) + P_j/2}]^2}, \end{aligned} \tag{12}$$

where $P_j \equiv \ln[\mu + \mathrm{i}\gamma 2(\zeta_j'' + \mathrm{i}\zeta_j')]$. The pulse polarizations $\boldsymbol{c}_{(j)}$ defined in a similar way as for the Manakov solitons are related to α_j, β_j by the formula

$$\mathrm{i}\boldsymbol{c}_{(j)} \equiv \mathrm{i}(c_{(j)1}, c_{(j)2}) = \frac{(\alpha_j, \beta_j)\mathrm{e}^{\mathrm{i}\,\mathrm{Im}\,\eta_{oj}}}{\sqrt{|\alpha_j|^2 + |\beta_j|^2}}. \tag{13}$$

We will show that (8) is an approximate solution of the MVNSE in a certain time window. The range of this time window is wider the smaller the differences between the velocities and width of the pulses are.

For $k_1 = k_2$, the solution (8) is relevant for $t \in (-\infty, \infty)$ but the pulses never collide under this condition. Considering the colliding pulses, one sees that the modules of the right-hand sides of (10) increase almost linearly with $|\Delta\zeta'|$ and with $|\Delta\zeta''|$, ($\Delta\zeta' \equiv \zeta_2' - \zeta_1'$, $\Delta\zeta'' \equiv \zeta_2'' - \zeta_1''$), for any fixed time point. However, at the final moment of the pulse collision $t = t_c$, ($t_c \sim \max\{|1/\zeta_1''|, |1/\zeta_2''|\}/|\Delta\zeta'|$ for $\Delta x_o \equiv x_{o2} - x_{o1} \approx 0, t = -t_c$ corresponds to the beginning moment of the collision), the modules of the right-hand sides of (10) increase exponentially with the ratio $(|\zeta_1'| + |\zeta_2'|)/|\Delta\zeta'|$. Taking this ratio as a constant ($|\Delta\zeta'| \sim (|\zeta_1'| + |\zeta_2'|)/2$), the modules of the right-hand sides of (10) can be thought of as linear in $|\Delta\zeta'|$ and $|\Delta\zeta''|$ ones. Then, for small $|\Delta\zeta'|, |\Delta\zeta''|$, and for small pulse velocities, $|\Delta\zeta'| \sim (|\zeta_1'| + |\zeta_2'|)/2$, (8) is the approximate solution to the MVNSE relevant to the collision regime of time $|t| < t_c$. Since separate (after the collision) pulses propagate almost independently (because the one-pulse solutions of (1) are exact), we evaluate the collision-induced polarization transform using the limit of the functions (8) with $t \to \infty$. At this limit

$$\begin{aligned} q_1 &\approx gf^*/f^2 = \mathrm{e}^{\delta_2}q^{(1)\prime} + \mathrm{e}^{\delta_1}q^{(2)\prime}, \\ q_2 &\approx hf^*/f^2 = \mathrm{e}^{\delta_2'}q^{(1)\prime} + \mathrm{e}^{\delta_1'}q^{(2)\prime}, \\ q^{(1)\prime} &= \mathrm{e}^{\eta_1}(\mu - \mathrm{i}\gamma k_2^*)\frac{\mathrm{e}^{R_2}(\mu - \mathrm{i}\gamma k_2^*) + \mathrm{e}^{\eta_1+\eta_1^*+R}(\mu - \mathrm{i}\gamma k_1^*)(\mu - \mathrm{i}\gamma k_2^*)}{[\mathrm{e}^{R_2}(\mu + \mathrm{i}\gamma k_2) + \mathrm{e}^{\eta_1+\eta_1^*+R}(\mu + \mathrm{i}\gamma k_1)(\mu + \mathrm{i}\gamma k_2)]^2}, \\ q^{(2)\prime} &= \mathrm{e}^{\eta_2}(\mu - \mathrm{i}\gamma k_1^*)\frac{\mathrm{e}^{R_1}(\mu - \mathrm{i}\gamma k_1^*) + \mathrm{e}^{\eta_2+\eta_2^*+R}(\mu - \mathrm{i}\gamma k_2^*)(\mu - \mathrm{i}\gamma k_1^*)}{[\mathrm{e}^{R_1}(\mu + \mathrm{i}\gamma k_1) + \mathrm{e}^{\eta_2+\eta_2^*+R}(\mu + \mathrm{i}\gamma k_2)(\mu + \mathrm{i}\gamma k_1)]^2}. \end{aligned} \tag{14}$$

Writing the envelopes of both the pulses of the above superposition in the form (12), one finds that the collision of two vector pulses characterized by the polarizations $\boldsymbol{c}_{(1)}, \boldsymbol{c}_{(2)}$ and by the wavenumbers ζ_1, ζ_2, respectively ($\zeta_j \equiv \zeta'_j + \mathrm{i}\zeta''_j$), results in the change of the polarizations of both the pulses according to the transformation

$$\mathrm{i}\boldsymbol{c}'_{(1)} = \left(\frac{\mu - \mathrm{i}\gamma k_2^*}{\mu + \mathrm{i}\gamma k_2}\right)^2 \frac{(\mathrm{e}^{\delta_2}, \mathrm{e}^{\delta'_2})\mathrm{e}^{\mathrm{i}\,\mathrm{Im}\,\eta_{o1}}}{\sqrt{|\mathrm{e}^{\delta_2}|^2 + |\mathrm{e}^{\delta'_2}|^2}} = \mathrm{e}^{\mathrm{i}\theta_1}\frac{\mathrm{i}}{\chi}\frac{\zeta_1^* - \zeta_2}{\zeta_1^* - \zeta_2^*}\left[\boldsymbol{c}_{(1)} + \frac{\zeta_2 - \zeta_2^*}{\zeta_2^* - \zeta_1^*}(\boldsymbol{c}_{(2)}^* \cdot \boldsymbol{c}_{(1)})\boldsymbol{c}_{(2)}\right],$$

$$\begin{aligned}\mathrm{i}\boldsymbol{c}'_{(2)} &= \left(\frac{\mu - \mathrm{i}\gamma k_1^*}{\mu + \mathrm{i}\gamma k_1}\right)^2 \frac{(\mathrm{e}^{\delta_1}, \mathrm{e}^{\delta'_1})\mathrm{e}^{\mathrm{i}\,\mathrm{Im}\,\eta_{o2}}}{\sqrt{|\mathrm{e}^{\delta_1}|^2 + |\mathrm{e}^{\delta'_1}|^2}} \\ &= \mathrm{e}^{\mathrm{i}\theta_2}\frac{\mathrm{i}}{\chi}\frac{\zeta_1^* - \zeta_2}{\zeta_1 - \zeta_2}\left[\boldsymbol{c}_{(2)} + \frac{\zeta_1 - \zeta_1^*}{\zeta_2 - \zeta_1}(\boldsymbol{c}_{(1)}^* \cdot \boldsymbol{c}_{(2)})\boldsymbol{c}_{(1)}\right],\end{aligned} \tag{15}$$

where χ is defined by (3). Up to the phase factor $\mathrm{e}^{\mathrm{i}\theta_{1(2)}} \equiv (\mu - \mathrm{i}\gamma k_{2(1)}^*)^2/(\mu + \mathrm{i}\gamma k_{2(1)})^2$, the transformed polarization is similar to that of the Manakov soliton.

For the small velocity difference, in the regime $|\Delta\zeta'| \ll |\zeta'_j|$, the pulse-interaction effects are known to play an important role in the soliton dynamics, especially for an imperfect (randomly birefringent) optical fiber. For the regime we consider, $|\Delta\zeta'| \sim (|\zeta'_1| + |\zeta'_2|)/2$, it is interesting if two differently polarized pulses interact stronger than identically-polarized ones. If it was so, the interaction effects (stopping or accelerating the relative motion of the pulses) could disturb our picture of the soliton-like pulse collisions while the identically-polarized pulses are known to collide like solitons of the MNSE. In order to determine the way in which the mutual polarization tilt of the pulses influences the strength of the interaction, we write the equations of motion of the pulse parameters following Keener, McLaughlin, Karpman and Solov'ev [17] who solved such equations perturbatively. We note that relevant equations for vector (Manakov) solitons [18] and for MNSE solitons [19] have been studied recently. Since the pulse collisions are slow, we are unable to detect collision-induced transformations of the pulse parameters solving their equations of motion because we cannot perform summation of all the terms of the perturbation expansions. Our aim is just to estimate if their changing is faster or slower than during the collision of MNSE solitons. Let us write the two-pulse solution of the MVNSE as

$$\boldsymbol{q} = \boldsymbol{q}^{(1)} + \delta\boldsymbol{q}^{(1)} + \boldsymbol{q}^{(2)} + \delta\boldsymbol{q}^{(2)}, \tag{16}$$

where $\boldsymbol{q}^{(j)} = (\alpha_j, \beta_j)q^{(j)}$ relate to the initial pulse envelopes while their deviations from the initial shape are connected to a slow change of the pulse parameters via

$$\begin{aligned}\frac{\delta\boldsymbol{q}^{(j)}}{\mathrm{d}t} = \mathrm{e}^{\mathrm{i}\Theta^{(j)}}&\left(\boldsymbol{q}'^{(j)}_{,\zeta'_j}\frac{\mathrm{d}\zeta'_j}{\mathrm{d}t} + \boldsymbol{q}'^{(j)}_{,x_{oj}}\frac{\mathrm{d}x_{oj}}{\mathrm{d}t} + \sum_{k=1}^{2}\boldsymbol{q}'^{(j)}_{,\varphi_{jk}}\frac{\mathrm{d}\varphi_{jk}}{\mathrm{d}t} + \sum_{k=1}^{2}\boldsymbol{q}'^{(j)}_{,\zeta''_{jk}}\frac{\mathrm{d}\zeta''_{jk}}{\mathrm{d}t}\right) \\ &+ \mathrm{i}\boldsymbol{q}^{(j)}\left(\Theta^{(j)}_{,\zeta'_j}\frac{\mathrm{d}\zeta'_j}{\mathrm{d}t} + \Theta^{(j)}_{,x_{oj}}\frac{\mathrm{d}x_{oj}}{\mathrm{d}t} + \sum_{k=1}^{2}\Theta^{(j)}_{,\zeta''_{jk}}\frac{\mathrm{d}\zeta''_{jk}}{\mathrm{d}t}\right).\end{aligned} \tag{17}$$

Here, the parameters ζ''_{jk} and φ_{jk} are defined by $\zeta''_{jk}\,\mathrm{e}^{\mathrm{i}\varphi_{jk}} \equiv c_{(j)k}\zeta''_j$, and $\boldsymbol{q}'^{(j)} = \boldsymbol{q}^{(j)} f^{(j)}/f^{(j)*} \equiv \boldsymbol{q}^{(j)}\,\mathrm{e}^{-\mathrm{i}\Theta^{(j)}}$. Since $\boldsymbol{q}^{(j)}$ are exact one-pulse solutions of the MVNSE, inserting $\boldsymbol{q}$ into (1), one finds the dynamical equations of $\delta\boldsymbol{q}^{(j)}$ up to the first order of perturbation

$$\begin{aligned}\mathrm{i}\frac{\delta\boldsymbol{q}^{(1)}}{\mathrm{d}t} &= -\left(\mu + \mathrm{i}\gamma\frac{\partial}{\partial x}\right)[(\boldsymbol{q}^{(1)} \cdot \boldsymbol{q}^{(2)*} + \boldsymbol{q}^{(1)*} \cdot \boldsymbol{q}^{(2)})\boldsymbol{q}^{(1)} + |\boldsymbol{q}^{(1)}|^2\boldsymbol{q}^{(2)}], \\ \mathrm{i}\frac{\delta\boldsymbol{q}^{(2)}}{\mathrm{d}t} &= -\left(\mu + \mathrm{i}\gamma\frac{\partial}{\partial x}\right)[(\boldsymbol{q}^{(2)} \cdot \boldsymbol{q}^{(1)*} + \boldsymbol{q}^{(2)*} \cdot \boldsymbol{q}^{(1)})\boldsymbol{q}^{(2)} + |\boldsymbol{q}^{(2)}|^2\boldsymbol{q}^{(1)}].\end{aligned} \tag{18}$$

Here the perturbation of the pulse parameters is due to the collision with the second pulse whose velocity and width are close to those of the first one. Following [17–19], using a set of orthogonal functions, one is able to find equations of motion of all the pulse parameters via projecting the above equations on some Hilbert-space directions. I simplify the original approach utilizing a proposal of Lai and Haus [20]. I find a 'canonical' set of functions (similar to that used in [21])

$$\tilde{q}^{(j)}_{k,x_{oj}} \equiv q'^{(j)}_{k,x_{oj}}, \qquad \tilde{q}^{(j)}_{k,\varphi_{jl}} \equiv q'^{(j)}_{k,\varphi_{jl}},$$
$$\tilde{q}^{(j)}_{k,\zeta'_j} \equiv q'^{(j)}_{k,\zeta'_j} - \left(2x_{oj} - \frac{1}{2\zeta''_j}\,\mathrm{Re}\,P_j + \frac{\mathrm{i}}{2}P_{j,\zeta'_j}\right) q'^{(j)}_{k,\varphi_{jk}} + \frac{1}{4\zeta''_j}P_{j,\zeta'_j}q'^{(j)}_{k,x_{oj}}, \tag{19}$$
$$\tilde{q}^{(j)}_{k,\zeta''_{jl}} \equiv q'^{(j)}_{k,\zeta''_{jl}} - \left(\frac{\zeta''_{jk}}{4\zeta''^3_j}\,\mathrm{Re}\,P_j - \frac{1}{4\zeta''_j}P_{j,\zeta''_{jl}}\right) q'^{(j)}_{k,x_{oj}} - \left(\frac{\mathrm{i}\delta_{kl}}{\zeta''_{jl}} + \frac{\mathrm{i}}{2}P_{j,\zeta''_{jl}}\right) q'^{(j)}_{k,\varphi_{jl}}$$

satisfying $\int_{-\infty}^{\infty}\sum_{k=1}^{2}\left[\tilde{q}^{(j)*}_{k,A}(x,t)\tilde{q}^{(j)}_{k,B}(x,t) - c.c.\right]\mathrm{d}x = 0$ for all the pairs of the pulse parameters (A,B) except (x_{oj},ζ'_j) and $(\varphi_{jk},\zeta''_{jl})$,

$$\mathrm{Im}\int_{-\infty}^{\infty}\sum_{k=1}^{2}\left[\tilde{q}^{(j)*}_{k,x_{oj}}(x,t)\tilde{q}^{(j)}_{k,\zeta'_j}(x,t) - c.c.\right]\mathrm{d}x = 16\mathrm{e}^{-\mathrm{Re}\,P_j}\,\mathrm{Im}\,P_j\,\mathrm{csec}(\mathrm{Im}\,P_j),$$
$$\mathrm{Im}\int_{-\infty}^{\infty}\sum_{k=1}^{2}\left[\tilde{q}^{(j)*}_{k,\varphi_{jl}}(x,t)\tilde{q}^{(j)}_{k,\zeta''_{jm}}(x,t) - c.c.\right]\mathrm{d}x = \frac{\zeta''^2_{jl}\zeta''_{jm}}{\zeta''^3_j}8\mathrm{e}^{-\mathrm{Re}\,P_j}\,\mathrm{Im}\,P_j\,\mathrm{csec}(\mathrm{Im}\,P_j), \tag{20}$$

and $\int_{-\infty}^{\infty}\sum_{k=1}^{2}\left[\tilde{q}^{(j)*}_{k,A}(x,t)q'^{(j)}_{k}(x,t) + c.c.\right]\mathrm{d}x = 0$ for all the pulse parameters A except ζ''_{jm},

$$\int_{-\infty}^{\infty}\sum_{k=1}^{2}\left[\tilde{q}^{(j)*}_{k,\zeta''_{jm}}(x,t)q'^{(j)}_{k}(x,t) + c.c.\right]\mathrm{d}x = -\frac{\zeta''_{jm}}{\zeta''_j}8\mathrm{e}^{-\mathrm{Re}\,P_j}\,\mathrm{Im}\,P_j\,\mathrm{csec}(\mathrm{Im}\,P_j). \tag{21}$$

Multiplying the left- and the right-hand sides of (18) by $\mathrm{e}^{-\mathrm{i}\Theta^{(j)}(x,t)}\tilde{\boldsymbol{q}}^{(j)*}_{,A}(x,t)$, $(A = x_{oj},\zeta'_j,\ldots)$, integrating the products over x, and using (20), (21), $\Theta^{(j)}_{,\varphi_{jl}} = 0$, one finds the equations of motion of the pulse parameters. As follows from the analysis of the right-hand sides of equations (18) the pulse parameters are changing with time faster the bigger value is taken by $|\boldsymbol{c}_{(1)}\cdot\boldsymbol{c}^*_{(2)}|$. Thus, the pulse-interaction strength decreases with deviating mutual pulse polarization tilt from the parallel one. In particular, let us note that considering the interaction of perpendicularly polarized pulses, $(\boldsymbol{c}_{(1)}\cdot\boldsymbol{c}^*_{(2)} = 0)$, the first-order interaction terms vanish and one has to include the terms of the second order of interaction on the right-hand sides of (18).

We complete our discussion of the dynamical equations of the ultra-short pulse propagation mentioning the role of the third-order dispersion effects which are completely neglected in (1). These effects become important in the case of fibers with a close to zero group velocity (vanishing second-order dispersion). However, adding the term $\mathrm{i}\beta q_{j,xxx}/3$ to the left-hand side of (1), one is able to find exact multi-soliton solutions of the resulting vector equation for $\beta = \gamma/\mu$, [22]. In [23], it has been shown that this relation ($\beta = \gamma/\mu$) holds for the ultra-short pulses within a certain microscopic model of the photon interactions in the fiber (the instantaneous Kerr interaction). Following [24], this microscopic model determines the pulse envelopes to be described with similar functions as those satisfying the NLSE. In the present work, I avoid restricting the considerations to any single quantum model.

It has been shown that, although the MVNSE solution cannot be written in the form of the Hirota (multi-soliton) expansion in general (except for the cases of identically polarized pulses or of a train of pulses with similar widths), the Hirota method enables one to find an asymptotic two-pulse solution. For some velocity difference and pulse-width difference regimes, these solutions are useful for describing the pulse collisions and they allow one to find the collision-induced transform of the pulse-polarization. The only consequence of including the nonlinear differential term into the pulse-envelope equation of motion on the collision-induced pulse polarization transform is an additional (compared to the Manakov transform for the VNLSE solitons) multiplication of the outcome polarizations by phase factors.

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